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# WEIGHT MINIMIZATION OF AXISYMMETRIC CLAMPED PLATES SUBJECT TO CONSTRAINTS

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## NOTATION

| | |
|---|---|
| $a$ | outer radius of plate |
| $\bar{B}_u, \bar{B}_L$ | power series functions of $\nu$, $\beta$ |
| $C$ | function of $\nu$, $\beta$, determined via the boundary conditions |
| $D$ | flexural rigidity depending on plate thickness, $h_x$ |
| $E$ | modulus of elasticity in tension and compression |
| $f(\beta)$ | power series function of $\nu$, $\beta$ |
| $G$ | undetermined multiplier function of $x$, $\beta$, $h_0$ |
| $G_m$ | maximum value of $G$ corresponding to $\beta_m$ at $x_m$ for first constraint |
| $h_a$ | plate thickness at outer radius, $a$ |
| $h_0$ | plate thickness at center of plate |
| $h_x$ | plate thickness at $x$ |
| $h_{0m}$ | value of $h_0$ which makes plate weight a minimum for first constraint |
| $h_{0mi}$ | value of $h_{0m}$ for the $i$th value of $\nu$ |
| $h_{01}, h_{02}, \ldots, h_{0i}$ | plate thickness varying at center of plate, see Fig. 2 |
| $k_0$ | constant, see equation (18) |
| $\bar{K}_1, \bar{K}_2, \bar{K}_3, \ldots$ | functions of $\nu$, $\beta$, see equations (16) and (17) |
| $M_r, M_t$ | bending moments/unit length |
| $m$ | subscript notation signifying a maximum or a minimum |
| $n$ | $n$th term, $n$th increment |
| $p$ | notation defined by equation (9) |
| $q$ | intensity of axisymmetric distributed load |
| $q_m$ | upper bound value of allowable load $q$ |
| $\bar{Q}_m$ | notation defined in equation (32) |
| $Q$ | shearing force parallel to $z$ axis/unit length of a section of a plate perpendicular to $r$ direction |
| $r$, $\theta$(or $t$), $z$ | cylindrical coordinates; $r$, $\theta$ in plane of plate |
| $w$ | displacement (deflection) in $z$ direction of middle-plane (surface) of a thin plate |
| $w_0$ | deflection at center of plate |
| $W$ | weight of plate |
| $x$ | dimensionless ratio $r/a$ |
| $x_m$ | value of $x$ which maximizes $G$ at $x_m$, using $\beta_m$ in equation (31), for first constraint |
| $x_1, x_2$ | limits of interval in which $G_m > 1$ |
| $y$ | dimensionless ratio $h_x/h_0$ |
| $\beta$ | dimensionless variable defining shape of plate |
| $\gamma$ | material density |
| $\beta_1, \beta_2, \ldots, \beta_n$ | "pencil of curves" for a given $h_{0i}$, see Fig. 3 |
| $\beta_m$ | value of $\beta$ satisfying at least the first constraint, making the plate weight an extremum |
| $\nu$ | Poisson's ratio |
| $\sigma_0$ | uniaxial yield stress, equal values in tension and compression |
| $\sigma_{0m}$ | lower bound value of the yield stress, $\sigma_0$, see equation (32) |
| $\sigma_r, \sigma_t$ | normal stresses in radial and tangential directions |
| $\phi$ | slope of middle surface as defined in equation (5) |

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| | |
|---|---|
| $\phi_h, \phi_p$ | homogeneous and particular solutions of differential equation of equilibrium, see equations (12) and (13) |
| $\phi_1$ | notation defined via equations (12) and (14) |
| $\phi', \phi''$ | first and second derivatives of $\phi$ with respect to $x$ |

## INTRODUCTION

THE importance of structural weight saving for aircraft, aerospace and deep-diving sea vehicles continues to give impetus to new methods and techniques of optimization. Shanley [1] and Gerard [2] have written extensively on this subject. The savings of even a few ounces of weight for the various structural components of a framework, adds up to many pounds which can then be put into useful payload or fuel. For some years now, many investigators have worked in the area of limit design to obtain the collapse loads of circular plates. These include Haythornthwaite [3], Freiberger and Tekinalp [4], Prager and Shield [5], Megarefs [6], Popov [7] and Save [8]. Reference [4] developed a minimum weight design for circular plates using von Mises' yield condition and the failure criterion of limit analysis. Limit analysis is significant in determining a more realistic factor of safety against total collapse or failure. For "one shot" operations, such as firing a non-manned vehicle to land permanently on the moon or Mars, the design to near collapse of certain portions of a structure may indeed save considerable weight. Of course, excessive displacements of certain portions of a structure may be undesirable insofar as this might affect the functioning of various black boxes on board. For space and sea vehicles which must be designed for reuse, such as long-time orbiting satellites or moon vehicles returned to Earth, very little of a structure can be permitted to yield. In fact, stresses and strains must remain linear throughout the useful life of the vehicle.

Little work has been done to date which imposes more than one constraint on a structural element. Haug [9], in 1966, developed a procedure for minimum weight design of beams with inequality constraints on stress and deflection using the calculus of variations. Saelman [10] considered strength and stiffness for wing box beams. Sherman [11] developed a procedure for the volume minimization of simply supported thin axisymmetric plates subject to constraints of stress and displacement.

This paper is concerned with the weight minimization of thin axisymmetric plates of variable thickness, clamped at the boundary, and subject to two constraint conditions. These are, (1) a specified maximum displacement, $w_0$, in terms of the outer radius, $a$, of the plate, in the form of an equality and, (2) the von Mises–Hencky yield criteria is not violated throughout the plate, i.e. the strain energy of deformation (distortion) is proportional to the uniaxial yield stress, $\sigma_0$, everywhere. Such a combination of constraints may be required for pressure-sensitive devices and where clearances are tight. Thin plate theory is considered valid. The state of stress in the plate remains elastic everywhere. The material is assumed to be homogeneous and isotropic.

To be meaningful in an engineering sense, the solution of this problem should be considered in terms of certain significant classes or diametral shapes of plates. Wang and Worley [12] optimized a class of thin shells of revolution according to shape. In this paper, the shape of the plate is such that the thickness varies exponentially with a single-sign curvature only (see Fig. 4). It is based on a form of Pichler's [13] exact solution of the differential equation of equilibrium for axisymmetric thin plates of varying thickness. Timoshenko [14] has summarized Pichler's findings. The plate thickness, $h_x$, in a diametral section, varies exponentially with the radius, according to the equation

$$y = h_x/h_0 = \exp(-\beta x)^2 \tag{1}$$

where $h_0$ is the thickness at the center of the plate. The necessary mathematics was developed to affect the minimization process satisfying the given constraints. The resulting equations, containing infinite power series, were solved on the IBM 360/50 digital computer. Solutions were obtained for the plate clamped at the boundary for a wide range of the variable constants, i.e. load, Poisson's ratio, modulus of elasticity, plate outer radius, yield stress and maximum displacement. Plotted curves and an example problem are included herein.

## FUNDAMENTAL RELATIONSHIPS

The differential equation of equilibrium of an element of a thin circular plate (see Fig. 1) is:

$$M_r+\frac{\mathrm{d}M_r}{\mathrm{d}r}\,.\,r-M_t+\tilde{Q}\,.\,r = 0 \tag{2}$$

where:

$$M_r = D\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}+\frac{v\phi}{r}\right) \tag{3}$$

$$M_t = D\left(\frac{\phi}{r}+v\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) \tag{4}$$

$$\phi = -\frac{\mathrm{d}w}{\mathrm{d}r} \tag{5}$$

$$\tilde{Q} = \frac{1}{2\pi r}\int_0^r q\,.\,2\pi r\,.\,\mathrm{d}r \tag{6}$$

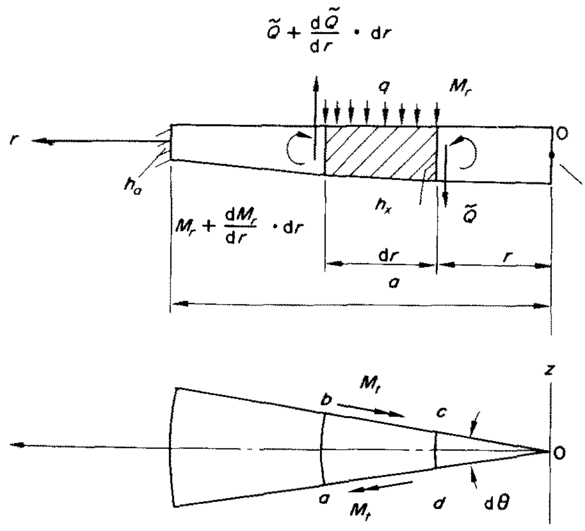


FIG. 1. Equilibrium of element *abcd* of a thin circular plate clamped on the boundary.

and,

$$D = \frac{Eh_x^3}{12(1-\nu^2)}. \tag{7}$$

The applied load, $q$, which can in general be an axisymmetric function of $r$, is taken as uniformly distributed in this paper.

The differential equation can be put in dimensionless form, by letting

$$x = \frac{r}{a} \tag{8}$$

$$p = \frac{6(1-\nu^2)a^3q}{Eh_0^3}. \tag{9}$$

The deflection of the middle surface (as noted in the introduction) of a thin plate of varying thickness is shown in Fig. 2, and defines $\phi$.

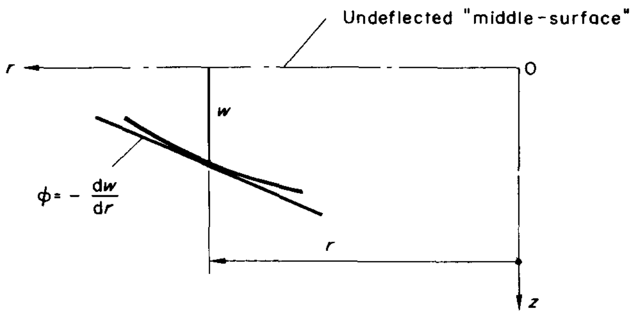


FIG. 2. Slope $\phi$ of the deflected middle surface at radius $r$.

Using equations (1)–(9), the differential equation of equilibrium of the second order with variable coefficients becomes

$$\phi'' + \left(\frac{1}{x} - 6\beta x\right) . \phi' - \left(\frac{1}{x^2} + 6\nu\beta\right) . \phi = -\frac{px}{y^3}. \tag{10}$$

The solution of equation (10) for $\phi$ is the sum of the homogeneous and particular solutions, $\phi_h$ and $\phi_P$, respectively, where

$$\phi = \phi_h + \phi_p = p\left[C\phi_1 - \frac{x}{6(3-\nu)\beta} \cdot \exp(3\beta x^2)\right] \tag{11}$$

and,

$$\phi_h = pC\phi_1 = pC\left[x+\frac{6\beta(1+\nu)}{2\cdot 4}\,.\,x^3+\frac{6^2\beta^2(1+\nu)(3+\nu)}{2\,.\,4\,.\,4\,.\,6}\,.\,x^5\right.$$
$$\left.+\frac{6^3\beta^3(1+\nu)(3+\nu)(5+\nu)}{2\,.\,4\,.\,4\,.\,6\,.\,6\,.\,8}\,.\,x^7\ldots\right] \tag{12}$$

$$\phi_p = -\frac{px}{6(3-\nu)\beta}\,.\,\exp(3\beta x^2). \tag{13}$$

Equation (11) is a uniformly convergent series, and may be differentiated or integrated term-by-term, as noted by Churchill [15]. For computer programming, equation (11) can be written in terms of $\bar{K}$'s, which are functions of $\nu$, $\beta$, only

$$\phi_h = pC[x+\bar{K}_1x^3+\bar{K}_2x^5+\bar{K}_3x^7+\ldots]. \tag{14}$$

The value of $C$ for the clamped plate is

$$C = \frac{1}{6(3-\nu)}\cdot\frac{1}{(1+\bar{K}_1+\bar{K}_2+\bar{K}_3+\ldots)}\cdot\frac{\exp(3\beta)}{\beta}. \tag{15}$$

The general expression for displacement, $w$, obtained from equations (5) and (11) is

$$w = \left\{apC\left[\left(\frac{1}{2}+\frac{\bar{K}_1}{4}+\frac{\bar{K}_2}{6}+\frac{\bar{K}_3}{8}+\ldots\right)-\left(\frac{x^2}{2}+\frac{\bar{K}_1}{4}x^4+\frac{\bar{K}_2}{6}x^6+\ldots\right)\right]\right.$$
$$\left.+\frac{ap}{6^2(3-\nu)}\cdot\frac{\exp(3\beta x^2)-\exp(3\beta)}{\beta^2}\right\}. \tag{16}$$

The maximum displacement occurs at $x = 0$, and is

$$w|_{x=0} = \left\{apC\left(\frac{1}{2}+\frac{\bar{K}_1}{4}+\frac{\bar{K}_2}{6}+\frac{\bar{K}_3}{8}+\ldots\right)+\frac{ap}{6^2(3-\nu)}\cdot\frac{[1-\exp(3\beta)]}{\beta^2}\right\}. \tag{17}$$

## CONSTRAINT CONDITIONS

The objective is to minimize the plate weight, while satisfying the two conditions of constraint, i.e.

$$w|_{x=0} = k_0 a \tag{18}$$

where $k_0$ is a specified coefficient and $a$ equals the outer radius of the plate. The von Mises–Hencky yield criteria is not violated throughout the plate, i.e.

$$\sigma_t^2+\sigma_r^2-\sigma_r\sigma_t \le \sigma_0^2 \tag{19}$$

where

$$\sigma_r = \frac{E}{2a(1-\nu^2)}\cdot\left(\phi'+\frac{\nu\phi}{x}\right)\,.\,h_x \tag{20}$$

$$\sigma_t = \frac{E}{2a(1-\nu^2)}\cdot\left(\frac{\phi}{x}+\nu\phi'\right)\,.\,h_x. \tag{21}$$

## MINIMIZATION OF WEIGHT SUBJECT TO CONSTRAINTS

In terms of the state variables, $h_0$, $\beta$, the weight $W$ of the plate is

$$W = \pi a^2 h_0 \frac{[1-\exp(-\beta)]}{\beta} \cdot \gamma. \tag{22}$$

In Fig. 3, for any $h_{0i}$, $i = 1, 2, \ldots, m$, a pencil of curves $\beta_1, \beta_2, \ldots, \beta_n$ can be drawn for the clamped plate, which literally map the field between $x = 0, 1$. Thus, for a given set of values of the variable constants $a$, $w_0$, $q$, $E$, $\nu$ and $\sigma_0$, the values of the variables $\beta$ and $h_0$ can be determined, in order to minimize the weight and satisfy the two constraints.

To start the minimization procedure, the weight can be minimized satisfying the first

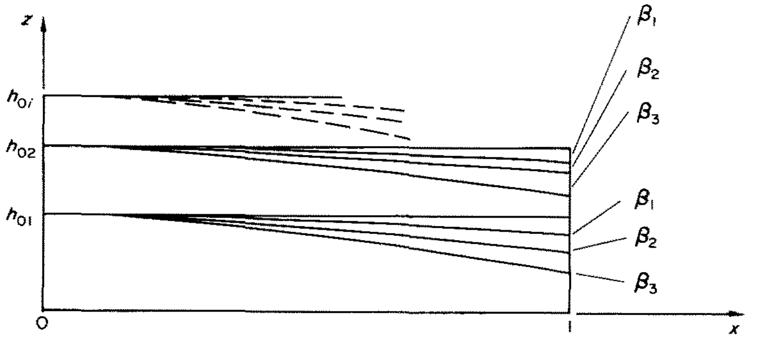

FIG. 3. Plate thickness as function of $h_0$ and $\beta$.

constraint only, i.e. equation (18). No attention is paid to the magnitudes of stress at this stage. If stress is not a consideration, the values of $\beta$ and $h_0$ obtained by minimizing the weight for the first constraint only, called $\beta_m$ and $h_{0m}$, respectively, are the most optimum values possible. Term $\beta_m$ is the upper bound extremum value of $\beta$. Following that, the second constraint, equation (19), is imposed upon the first solution. The values of $\beta_m$ and $h_{0m}$ may satisfy both constraints when the magnitudes of $\sigma_0$, $\nu$ and $(q^{\frac{1}{4}}k_0E)$ are within certain limits (see Figs. 5–7). Beyond these limits, the values of $\beta$ and $h_0$ are either different from $\beta_m$ and $h_{0m}$, respectively, thereby yielding a lesser optimum weight, or else no solution exists at all.

The shape of the plate is defined by equation (1), for which positive values of $\beta$ yield convex diametral surfaces, while negative $\beta$ yields concave surfaces (see Fig. 4).

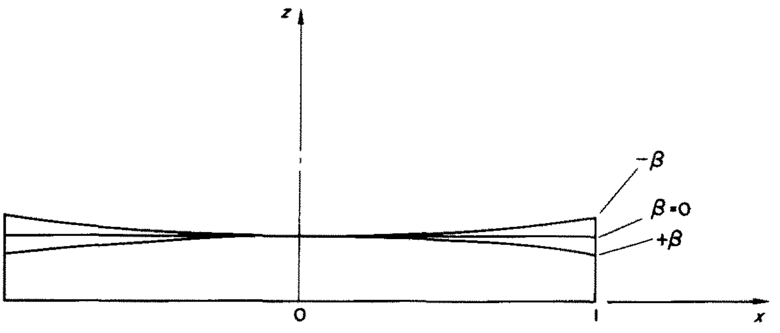

FIG. 4. Diametral shape of plate as function of $\beta$.

## WEIGHT MINIMIZED FOR THE FIRST (DISPLACEMENT) CONSTRAINT ONLY

For the clamped plate, equation (17) can be written as:

$$w|_{x=0} = k_0 a = \frac{(1-v^2)}{(3-v)} \cdot \frac{a^4 q}{E} \cdot \frac{1}{h_0^3} \cdot f(\beta) \tag{23}$$

where

$$f(\beta) = \frac{\exp(3\beta)}{\beta} \cdot \frac{\bar{B}_u}{\bar{B}_L} + \frac{[1-\exp(3\beta)]}{6\beta^2} \tag{24}$$

where

$$\bar{B}_u = \frac{1}{2} + \frac{\bar{K}_1}{4} + \frac{\bar{K}_2}{6} + \frac{\bar{K}_3}{8} + \ldots \tag{25}$$

$$\bar{B}_L = 1 + \bar{K}_1 + \bar{K}_2 + \bar{K}_3 + \ldots . \tag{26}$$

From equations (22) and (23), the plate weight can be written as

$$W = \pi a^3 \cdot \left[\frac{(1-v^2)}{(3-v)}\right]^{\frac{1}{3}} \cdot \frac{1}{(k_0 E/q)^{\frac{1}{3}}} \cdot \frac{[1-\exp(-\beta)]}{\beta} \cdot [f(\beta)]^{\frac{1}{3}} \cdot \gamma. \tag{27}$$

For minimum weight

$$\frac{\partial W}{\partial \beta} = \frac{\partial}{\partial \beta}\left\{\frac{[1-\exp(-\beta)]}{\beta} \cdot [f(\beta)]^{\frac{1}{3}}\right\} = 0. \tag{28}$$

The solution of equation (28) requires the determination of the convergence values of four infinite series, i.e. $\bar{B}_u$, $\bar{B}_L$, $\partial \bar{B}_u/\partial \beta$ and $\partial \bar{B}_L/\partial \beta$. For the clamped plate, equation (28), expanded, becomes

$$\tfrac{1}{3} \cdot [1-\exp(-\beta)] \cdot \frac{\left[\frac{\bar{B}_u}{\bar{B}_L} \cdot \beta^2 \cdot e^{3\beta}\left(3 - \frac{1}{\beta} - \frac{1}{\bar{B}_L} \cdot \frac{\partial \bar{B}_L}{\partial \beta}\right) + \beta^2 \cdot e^{3\beta} \cdot \frac{1}{\bar{B}_L} \cdot \frac{\partial \bar{B}_u}{\partial \beta} - \frac{\beta}{2} \cdot \exp(3\beta) - \tfrac{1}{3} \cdot [1-\exp(3\beta)]\right]}{\left\{\beta \cdot [\exp(3\beta)] \cdot \frac{\bar{B}_u}{\bar{B}_L} + \frac{1}{6} \cdot [1-\exp(3\beta)]\right\}}$$

$$+(1+\beta) \cdot [\exp(-\beta)] - 1 = 0. \tag{29}$$

The extremum values, $\beta_m$, were obtained for six values of $v$, and are listed in Table 1. From equation (23), corresponding values of $h_0 \equiv h_{0m}$ can then be computed. The solution of equation (29) is dependent only on $v$ for the first constraint only.

TABLE 1. $\beta_m$ FOR VARIOUS VALUES OF $v$—CLAMPED PLATE

| $v$ | 0·15 | 0·20 | 0·25 | 0·30 | 1/3 | 0·40 |
|---|---|---|---|---|---|---|
| $\beta_m$ | −0·0826 | −0·0150 | 0·1175 | 0·2968 | 0·5177 | 0·8808 |

## WEIGHT MINIMIZED FOR THE FIRST (DISPLACEMENT) AND SECOND (STRESS) CONSTRAINTS

The second constraint condition can be handled by rewriting equation (19) with the equality sign, and introducing an undetermined multiplier, $G = G(x, \beta, h_0)$

$$\sigma_t^2 + \sigma_r^2 - \sigma_r \sigma_t = G \,.\, \sigma_0^2 \tag{30}$$

in which $G \leq 1$ for all $x$. To determine the value of $x \equiv x_m$ which maximizes $G \equiv G_m$ at $x_m$, $\beta_m$ from the first constraint and equations (20) and (21) are substituted into equation (30). The maximum allowable $q \equiv q_m$ requires that $G_m = 1$ at $x_m$, and $G \leq 1$ for all $x$. The critical root $x_m$ is determined from

$$\frac{\partial G}{\partial x} = 0 \tag{31}$$

(see Ref. [11, pp. 746, 747]). The maximum strain energy of deformation occurs at $x_m$, the critical root of equation (31). $x_m = 1{\cdot}0$ for all values of $v$ and corresponding values of $\beta_m$ given in Table 1. Other roots of equation (31) exist at $x = 0{\cdot}0$ for all values of $v$ and at $x \simeq 0{\cdot}55$–$0{\cdot}80$ for $v = 0{\cdot}15$–$0{\cdot}40$, but these roots do not yield minimum weights.

## OPTIMUM WEIGHT SOLUTION SATISFYING LOWER BOUND VALUE OF YIELD STRESS, $\sigma_{0m}$

With the use of equation (23), equation (30) can be expressed in terms of state variables $x$ and $\beta$ only. Letting $G = 1$, then for a given value of $v$ and corresponding values of $\beta_m$ and $x_m$, equation (30) can be put in the form

$$(9)^{\frac{3}{4}} \cdot \frac{(3-v)}{(1-v^2)} \cdot (q^{\frac{1}{2}} \,.\, k_0 \,.\, E) \,.\, \frac{1}{f(\beta_m)} \cdot \bar{Q}_m^{\frac{3}{4}} \,.\, \exp(-\tfrac{3}{2}\beta_m \,.\, x_m^2) = (1) \,.\, \sigma_{0m}^{\frac{3}{2}} \tag{32}$$

where $\bar{Q}_m = \bar{Q}_m(x_m, \beta_m, h_{0m})$. Lower bound values of $\sigma_{0m}$ can be obtained for corresponding values of the function $(q^{\frac{1}{2}} k_0 E)$. As long as the given $\sigma_0 \geq \sigma_{0m}$, both constraints are satisfied, and the weight is minimized at its optimum (see Fig. 5).

## OPTIMAL WEIGHT SOLUTION WHEN $\sigma_0 < \sigma_{0m}$

Consider a given set of values $q$, $w_0$, $a$, for some set of values $E$, $v$ and $\sigma_0$. If the function $(q^{\frac{1}{2}} k_0 E)$ yields a lower bound $\sigma_{0m} > \sigma_0$, the second constraint is not satisfied. Then, there exists an interval $x_1 < x < x_2$ where $G > 1$. If a penalty on the optimum weight is acceptable and providing $\sigma_0/\sigma_{0m}$ is not less than a certain value determined via the computer process, both constraints can be satisfied. The results are plotted in Fig. 7. A detailed procedure to obtain a computer solution is given in Ref. [11], and requires certain multistage decision processes.

## EXAMPLE PROBLEM SOLVED USING PLOTTED CURVES OBTAINED FROM THE COMPUTER SOLUTION

The derived equations were solved on the IBM 360/50 computer using Fortran IV, and the results were plotted in Figs. 5–8. The solution to the problem of weight minimization

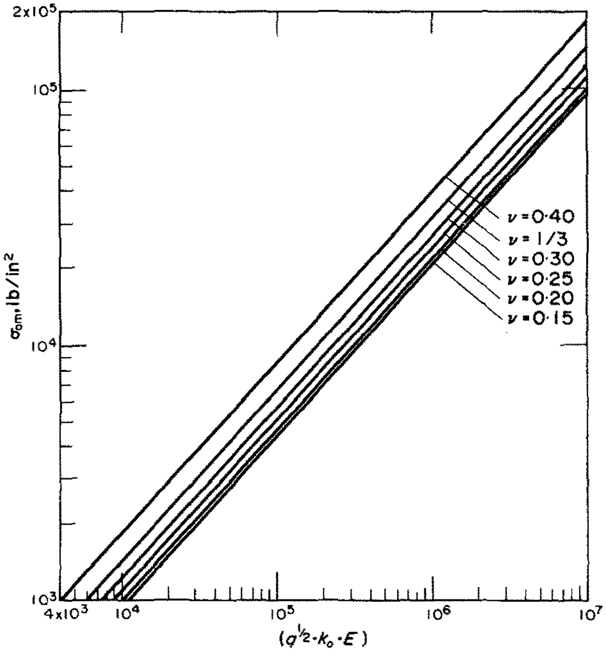


FIG. 5. $\sigma_m$ vs. $(q^{\frac{1}{2}} . k_0 . E)$ for variable $\nu$—clamped plate.

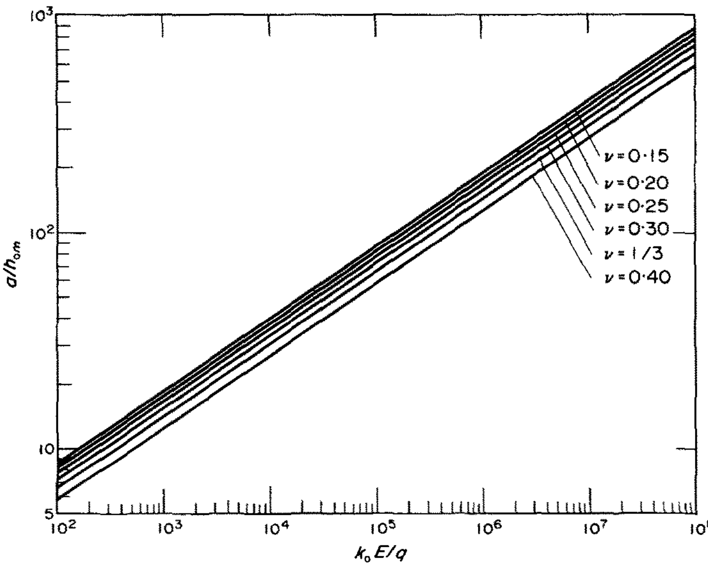


FIG. 6. $a/h_{0m}$ vs. $k_0E/q$ for variable $\nu$—clamped plate.

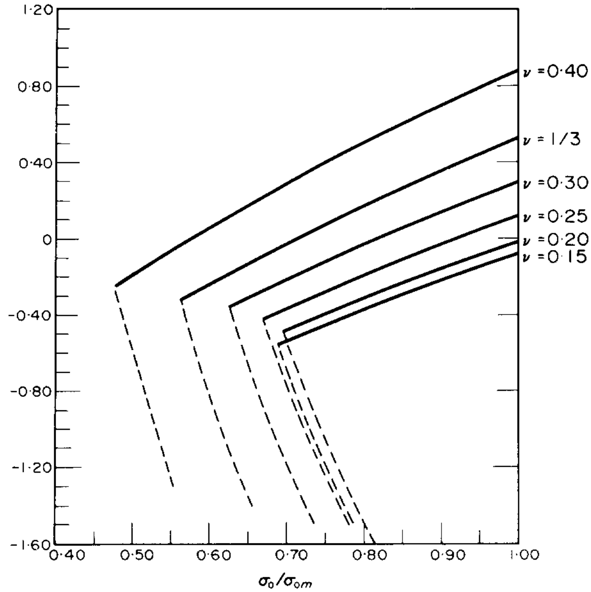


FIG. 7. $\beta$ vs. $\sigma_0/\sigma_{0m}$ for variable $\nu$—clamped plate.

of a thin plate of variable thickness can be put into one of two categories, or cases, in the search for a optimum solution.

*Case* 1

Find $h_0$, $\beta$ and allowable load $q \equiv q_m$, when given quantities $a$, $w_0$, $\nu$, $E$ and $\sigma_0$.

For this case the weight can be optimized at its extremum value, in which $\beta \equiv \beta_m$, $h_0 \equiv h_{0m}$, for $G \leq 1$ for all $x$ and $G_m = 1$ at $x_m$. This yields the upper bound value of the

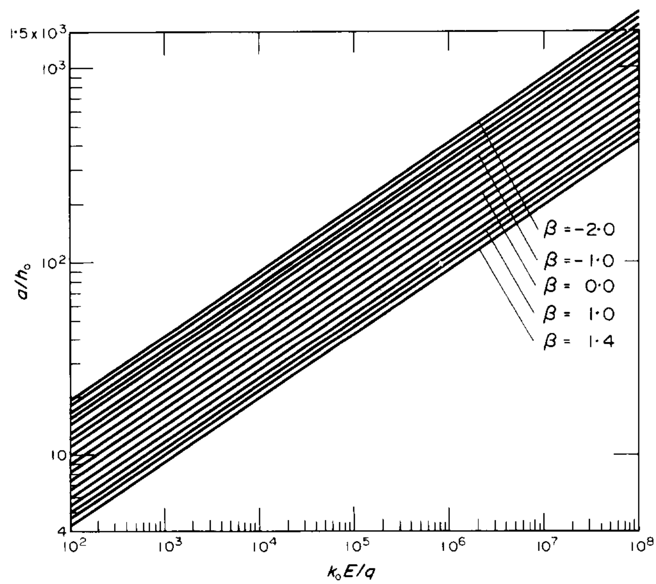


FIG. 8a. $a/h_0$ vs. $k_0E/q$ for $\nu = 0{\cdot}15$ and variable $\beta$—clamped plate.

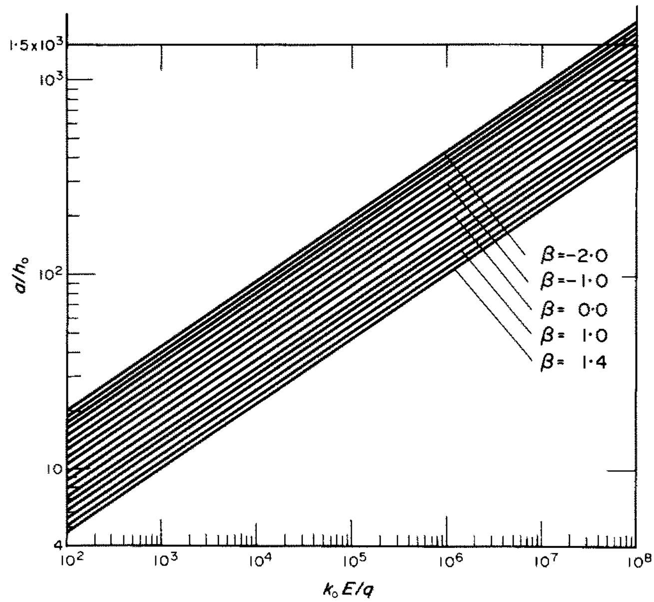


FIG. 8b. $a/h_0$ vs. $k_0E/q$ for $\nu = 0{\cdot}40$ and variable $\beta$—clamped plate.

allowable load, $q \equiv q_m$, by setting $\sigma_0 \equiv \sigma_{0m}$, and thus satisfies the first and second constraints.

*Example* 1. Given $a = 10$ in., $w_0 = 0{\cdot}07$ in., $E = 3\times10^7$ psi, $\nu = \frac{1}{3}$ and $\sigma_0 = 30{,}000$ psi. From Table 1, $\beta_m = 0{\cdot}5177$; from Fig. 5, letting $\sigma_0 \equiv \sigma_{0m}$, then $(q^{\frac{1}{4}}k_0E) = 9{\cdot}40\times10^5$; using equation (17), $q \equiv q_m = 20{\cdot}0$ psi; then, using Fig. 6 which plots $k_0E/q$ against $a/h_{0m}$, $k_0E/q = 1{\cdot}05\times10^4$ and $a/h_{0m} = 31{\cdot}6$ for $\nu = \frac{1}{3}$; thus, $h_{0m} = 0{\cdot}32$ in. and from equation (1), $h_a = 0{\cdot}19$ in.; from equation (22), for a steel plate, $W = 0{\cdot}283\times77{\cdot}4 = 21{\cdot}9$ lb.

*Case* 2

Find $h_0$, $\beta$, when given quantities $a$, $w_0$, $\nu$, $E$, $q$ and $\sigma_0$.

When $\sigma_0 < \sigma_{0m}$, as determined from Fig. 5, then Case 2 applies. For this case, the minimum weight is attained which satisfies both constraints.

*Example* 2. (Refer to data in example 1.) Given $a = 10$ in., $w_0 = 0{\cdot}07$ in., $E = 3\times10^7$ psi, $\nu = \frac{1}{3}$, $q = 21{\cdot}4$ psi and $\sigma_0 = 18{,}000$ psi. Since $(q^{\frac{1}{4}}k_0E) = 9{\cdot}40\times10^5$ and $\sigma_{0m} = 30{,}000$ psi from Fig. 5, then $\sigma_0 < \sigma_{0m}$ and Case 2 applies. Now, $\sigma_0/\sigma_{0m} = 0{\cdot}60$ and in Fig. 7, $\beta = -0{\cdot}243$. Then, in Figs. 8, for $k_0E/q = 1{\cdot}05\times10^4$, $a/h_0 = 44{\cdot}6$ for $\nu = \frac{1}{3}$ and $h_0 = 0{\cdot}22$ in. From equation (1), $h_a = 0{\cdot}28$ in. From equation (22), $W = 0{\cdot}283\times79{\cdot}8 = 22{\cdot}6$ lb. We note in Fig. 7, that had $\sigma_0$ been taken as less than 16,000 psi, no solution would exist.

## SUMMARY AND CONCLUSIONS

Clamped axisymmetric thin plates of varying thickness can be minimized for weight subject to both displacement and stress constraints. These constraints specify a given displacement at the center of the plate and do not violate the von Mises–Hencky yield criteria. The material is assumed homogeneous and isotropic. The state of stress remains everywhere elastic. Such a combination of constraints may be required for pressure-sensitive devices, and where clearances are tight.

The necessary mathematics was developed to accomplish the minimization process, and the resulting equations, containing infinite power series, were solved on the IBM 360/50 digital computer. For a wide range of the variable constants, i.e. $a$, $w_0$, $q$, $E$, $\nu$ and $\sigma_0$, curves were plotted which readily yield an optimum diametral shape. The extent, to which the plate weight may be optimized, depends on the relative magnitude of these quantities. No solution exists for certain ranges of these values, as is evident in the plotted curves.

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